

Exchange Relation in sl_3 WZNW model in Semiclassical Limit

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Abstract

We consider the exchange relations of screened vertex operators in the sl_3 Wess-Zumino-Novikov-Witten(WZNW) model in the semiclassical limit (where level k tends to infinity). We demonstrate that the coefficients of the exchange relations of the screened vertex operators coincide with the ones of the spider diagrams[1, 2]. The spider diagrams are composed of 3-point vertices, and differ from ordinary string diagrams.

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1 introduction

The WZNW model has been well studied through its relationship with other mathematical models[3, 4, 5]. As a recent topic, there are the studies of the relations between the Racah matrices and the HOMFLY polynomials[6, 7, 8, 9, 10, 11, 12].

Especially in the case of sl_2 WZNW model, as for its exchange relations, the coefficients have been computed explicitly for screened vertex operators possessing arbitrary weights. The coefficients are in accordance with the Racah-Wigner q -6j-symbol[13, 14, 15, 16, 17, 18, 19].

On the other hand, in the case of sl_3 WZNW model, its exchange relations have not been studied very well. When the rank of the group is two or more than two, the free field representation (or so-called Wakimoto representation[20, 21]) possesses multiple number of bosonic fields. And if we give a screened vertex operator including at least two kinds of screening charges, an ambiguity in ordering of its screening charges appears.

To begin with, one has to introduce six appropriate screened vertex operators, as shown in (22), so that their exchange relation takes a closed form. Then the coefficients of the exchange relations coincide with the exchange relations of spider diagrams introduced in [1, 2]. The spider diagram is a useful method to represent $U_q(sl_3)$ in a graphical way. In this paper, we consider its semiclassical structure given by the limit where q equals 1. We do not know any systematic way of handling this procedure in the case where q is a root of unity.

The organization of the paper is as follows. In section 2, first we introduce the sl_3 WZNW model in Wakimoto representation and define the vertex operators, the screening charges and the six appropriate screened vertex operators. In section 2.3, we demonstrate the contour integrals of the screening charges, and then find out the useful identities. In section 2.4, we list the results of calculations of exchange relations. In section 3, we summarize the spider diagram and its exchange relations. In the appendix A, we show concrete calculations of the spider diagram.

2 WZNW model

In this paper, we are concerned about the semiclassical limit of the exchange relation only. The semiclassical limit implies the infinity limit in level k of the sl_3 WZNW model. Even within this simplification we still obtain a nontrivial exchange relation.

2.1 definitions of free fields

Latin letters take 1, 2, while greek letters take 1, 2, 3.

It is well known that the affine Lie current of the sl_3 Lie algebra can be constructed by using the several free fields. As a introduction to WZNW model, there are references [4, 22, 23]

We introduce free fields by the following formal Laurent series.

$$\phi_i(z) = \frac{1}{\kappa} \left(q_i + (a_i)_0 \log z - \sum_{n \neq 0} \frac{(a_i)_n}{n} z^{-n} \right), \quad (1)$$

$$a_i(z) = \sum_{n \in \mathbb{Z}} (a_i)_n z^{-n-1}, \quad (2)$$

$$\beta_\alpha(z) = \sum_{n \in \mathbb{Z}} (\beta_\alpha)_n z^{-n-1}, \quad (3)$$

$$\gamma_\alpha(z) = \sum_{n \in \mathbb{Z}} (\gamma_\alpha)_n z^{-n}. \quad (4)$$

Here $\kappa = k + 3$, and q_i , $(a_i)_n$, $(\beta_\alpha)_n$, and $(\gamma_\alpha)_n$ obey the following commutation relations.

$$[(a_i)_m, (a_j)_n] = \kappa m a_{ij} \delta_{m+n,0}, \quad (5)$$

$$[(a_i)_0, q_j] = \kappa a_{ij}, \quad (6)$$

$$[(\beta_\alpha)_m, (\gamma_\beta)_n] = \delta_{\alpha\beta} \delta_{m+n,0} \quad (7)$$

$\delta_{m,n}$ is Kronecker delta, and a_{ij} is Cartan matrix of sl_3 :

$$a_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}. \quad (8)$$

Vacuum expectation value of an operator is written as

$$\langle \mathcal{O} \rangle = \langle 0 | \mathcal{O} | 0 \rangle \quad (9)$$

The bracket is defined to satisfy

$$\langle 0 | \{ (a_i)_{n<0}, q_i, (\beta_\alpha)_{n<0}, (\gamma_\alpha)_{n \geq 0} \} = 0, \quad \{ (a_i)_{n \geq 0}, (\beta_\alpha)_{n \geq 0}, (\gamma_\alpha)_{n<0} \} | 0 \rangle = 0, \quad \langle 0 | 1 | 0 \rangle = 1. \quad (10)$$

The operator product expansions of the above free fields are

$$\phi_i(z)\phi_j(w) = \frac{a_{ij}}{\kappa} \log(z-w) + \cdots, \quad (11)$$

$$a_i(z)\phi_j(w) = \frac{a_{ij}}{z-w} + \cdots, \quad (12)$$

$$\phi_i(z)a_j(w) = -\frac{a_{ij}}{z-w} + \cdots, \quad (13)$$

$$\beta_\alpha(z)\gamma_\beta(w) = \frac{\delta_{\alpha\beta}}{z-w} + \cdots. \quad (14)$$

Here the terms omitted are the normal ordered terms which do not have poles at $z = w$.

Two vertex operators are introduced as an exponential function of the free fields ϕ_i .

$$V_1(z) =: e^{\frac{1}{3}(2\phi_1(z)+\phi_2(z))} : , \quad (15)$$

$$V_2(z) =: e^{\frac{1}{3}(\phi_1(z)+2\phi_2(z))} : . \quad (16)$$

By Wick theorem, they show singularity.

$$: e^{\phi_i(z)} :: e^{\phi_i(w)} := (z-w)^{a_{ij}/\kappa} : e^{\phi_i(z)} e^{\phi_i(w)} : \quad (17)$$

2.2 definitions of bases and exchange relations

Usually the screening charges are defined as following.

$$\tilde{Q}_1(z) \equiv \frac{1}{1-q^{-1}} \oint_{\Gamma_z} (\beta_1(w) + \frac{1}{2}\gamma_2(w)\beta_3(w))e^{-\phi_1(w)}dw, \quad (18)$$

$$\tilde{Q}_2(z) \equiv \frac{1}{1-q^{-1}} \oint_{\Gamma_z} (\beta_2(w) - \frac{1}{2}\gamma_1(w)\beta_3(w))e^{-\phi_2(w)}dw. \quad (19)$$

Here $q = e^{2\pi i/\kappa}$, and the integration contour Γ_z starts with z , ends with z , and surrounds ordered vertex operators to its right.

To simplify the calculation of the contour integrals, we use the screening charges which have no $\beta_\alpha(z)$ and $\gamma_\alpha(z)$. This simplification does not change the coefficients of exchange relations.

$$Q_i(z) \equiv \frac{1}{1-q^{-1}} \oint_{\Gamma_z} s_i(z)dw, \quad s_i(z) \equiv e^{-\phi_i(z)}. \quad (20)$$

While the factor $\frac{1}{1-q^{-1}}$ diverges to infinity when we take the classical limit ($q \rightarrow 1$), the screening charge Q_i is finite thank to an infinitesimal factor comes from the contour integrals like as (38).

When we multiply by a number of screening charges

$$Q_2(z_3)Q_1(z_2)Q_1(z_1)\cdots, \quad (21)$$

these contours take the form as shown in Fig.1.

In the semiclassical limit $\kappa \rightarrow \infty$ ($q \rightarrow 1$), it is immaterial what branch we choose, because the factor q^r (r is a rational number) originating in branch becomes 1 in this limit.

The following equations introduce the six screened vertex operators, whose exchange relations are

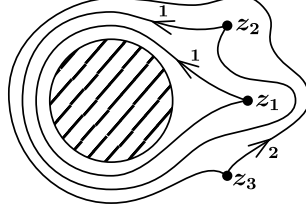


Figure 1: **contours:** The labels 1 and 2 near the arrows denote the subscripts of each screening charge. The central disk covers the vertex operators on the right hand of the three screening charges $Q_2(z_3)Q_1(z_2)Q_1(z_1)$.

take a closed form.

$$\begin{aligned}
 U^{(1,0)}(z) &\equiv V_1(z) , \\
 U^{(-1,1)}(z) &\equiv V_1(z)Q_1(z)\frac{1}{J_1} , \\
 U^{(0,-1)}(z) &\equiv V_1(z)\left(Q_1(z)Q_2(z)(J_2+1) - Q_2(z)Q_1(z)J_2\right)\frac{1}{J_2(J_1+J_2+1)} , \\
 &= V_1(z)Q_{12}(z)\frac{1}{J_2(J_1+J_2+1)} , \\
 U^{(0,1)}(z) &\equiv V_2(z) , \\
 U^{(1,-1)}(z) &\equiv V_2(z)Q_2(z)\frac{1}{J_2} , \\
 U^{(-1,0)}(z) &\equiv V_1(z)\left(Q_2(z)Q_1(z)(J_1+1) - Q_1(z)Q_2(z)J_1\right)\frac{1}{J_1(J_1+J_2+1)} , \\
 &= V_2(z)Q_{21}(z)\frac{1}{J_1(J_1+J_2+1)} .
 \end{aligned} \tag{22}$$

Here the superscripts of U are the weights of each operator respectively, and

$$Q_{i\bar{i}}(z) \equiv \frac{1}{(1-q^{-1})^2} \oint_{\Gamma_z} dw_1 s_i(w_1) \oint_{\Gamma_{w_1}} dw_2 s_{\bar{i}}(w_2), \tag{23}$$

where $i = 1, 2, \bar{1} \equiv 2$ and $\bar{2} \equiv 1$, and the operators $J_i \equiv (a_i)_0$ count the number of the vertex operators to its right.

$$\begin{aligned}
 [J_i, V_j(z)] &= \delta_{ij} V_1(z) , \\
 [J_i, U^\lambda(z)] &= \lambda_i U^\lambda(z) ,
 \end{aligned} \tag{24}$$

Here and below, the indices λ_i ($i = 1, 2, 3, 4$) run over the weights $\{(1, 0), (-1, 1), (0, -1), (0, 1), (1, -1), (-1, 0)\}$.

We define $|\mu\rangle$ which is an eigenvector of J_i .

$$J_1|\mu\rangle = m|\mu\rangle , \quad J_2|\mu\rangle = n|\mu\rangle . \tag{25}$$

here $\mu = (m, n)$ and

$$|\mu\rangle \equiv \prod_{i=1}^m V_1(x_i) \prod_{j=1}^n V_2(y_j) |0\rangle , \tag{26}$$

We consider x_i and y_j as arbitrary positions.

The exchange relations between screened vertex operators (22) are expressed as

$$U^{\lambda_1}(z)U^{\lambda_2}(w)|\mu\rangle = \sum_{\lambda_3\lambda_4} [c_{WZNW}(\mu)]_{\lambda_3\lambda_4}^{\lambda_1\lambda_2} U^{\lambda_3}(w)U^{\lambda_4}(z)|\mu\rangle. \quad (27)$$

In the section 2.4, we will list its concrete expression.

2.3 contour integrals

In this section, we show some identities of the contour integrals carried out in the semiclassical limit $\kappa \rightarrow \infty$. As a reference for the contour integrals, we have used [24].

In this section, we confirm the *identities* listed in (28) to (35). These identities are used to calculate the exchange relations. Commutativity:

$$[Q_i(z), V_{\bar{i}}(z_1)] = 0, \quad (28)$$

$$[Q_i(z), V_{\bar{i}}(z_1, z_2)] = 0, \quad (29)$$

$$[Q_i(z), I(z_1, z_2)] = 0, \quad (30)$$

$$[Q_i(z), I_j(z_1, z_2, z_3)] = 0, \quad (31)$$

Symmetry:

$$V_i(z_1, z_2) = -V_i(z_2, z_1), \quad (32)$$

$$I(z_1, z_2) = I(z_2, z_1), \quad (33)$$

$$I_i(z_1, z_2, z_3) = I_i(z_2, z_3, z_1) = -I_i(z_2, z_1, z_3), \quad (34)$$

$$I_i(z_1, z_2, z_3)V_{\bar{i}}(z_4) + I(z_1, z_4)V_{\bar{i}}(z_2, z_3) + I(z_2, z_4)V_{\bar{i}}(z_3, z_1) + I(z_3, z_4)V_{\bar{i}}(z_1, z_2) = 0, \quad (35)$$

where $i, j = 1, 2$, $\bar{1} \equiv 2$, $\bar{2} \equiv 1$ and

$$\begin{aligned} V_{\bar{i}}(z_1, z_2) &:= [V_i(z_1)Q_i(z_1), V_i(z_2)], \\ I(z_1, z_2) &:= [V_1(z_1)Q_1(z_1), [Q_2(z_1), V_2(z_2)]], \\ I_i(z_1, z_2, z_3) &:= [V_i(z_1)Q_i(z_1), [Q_{\bar{i}}(z_1), [V_i(z_2)Q_i(z_2), V_i(z_3)]]]. \end{aligned} \quad (36)$$

To express the contour integrals by using a graphical way, we list the rules of the diagrams as followings.

- The points denote the positions of vertex operators.
- The labels 1 and 2 near the points, denote the indices of the vertex operators.
- The lines denote the contours.
- The arrows denote the directions of integrals.
- The labels 1 and 2 near the arrows, denote the indices of the screening charges.

From now on, let us carry out the calculation for the above identities.

The first commutation relation as below often appears in the actual calculations of the exchange relations.

$$V_2(z, 0) = [V_1(z)Q_1(z), V_1(0)] = \left[V_1(z) \left(\frac{1}{1-q^{-1}} \oint_{\Gamma_z} dw s_1(w) \right), V_1(0) \right]. \quad (37)$$

Since the κ is large enough to converge the integral around the pole, one can take the contour in accordance with Cauchy's integral theorem as following.

$$\begin{array}{c} 1 \\ \curvearrowright \\ 1 \bullet \\ 0 \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} = \begin{array}{c} 1 \\ \curvearrowright \\ 1 \bullet \\ 0 \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} = (1 - q^{-1}) \begin{array}{c} 1 \\ \curvearrowright \\ 1 \bullet \\ 0 \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} \quad (38)$$

While the factor $1 - q^{-1}$ is infinitesimal, there is factor $1/(1 - q^{-1})$ in the definitions of screening charges (20), so that the contour integral (37) becomes finite.

Next, we consider the following commutation relation.

$$[Q_1(z_2), V_2(z, 0)] = 0, \quad (39)$$

The contour of $V_2(z, 0)$ is similar to (38). By adding the contour coming from $Q_1(z_2)$, we obtain following diagram.

$$\begin{array}{c} 1 \\ \curvearrowright \\ 1 \bullet \\ 0 \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z_2 \end{array} = \begin{array}{c} 1 \\ \curvearrowright \\ 1 \bullet \\ 0 \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z_2 \end{array} = \begin{array}{c} 1 \\ \curvearrowright \\ 1 \bullet \\ 0 \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z_2 \end{array} = 0 \quad (40)$$

The second equality is shown by cutting off the contour into four contours as following.

$$\begin{array}{c} 1 \\ \curvearrowright \\ 1 \bullet \\ 0 \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z_2 \end{array} = \begin{array}{c} 1 \\ \curvearrowright \\ 1 \bullet \\ 0 \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} + q \begin{array}{c} 1 \\ \curvearrowright \\ 1 \bullet \\ 0 \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} \\ + \begin{array}{c} 1 \\ \curvearrowright \\ 1 \bullet \\ 0 \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} + q \begin{array}{c} 1 \\ \curvearrowright \\ 1 \bullet \\ 0 \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} = 0 \quad (41)$$

Hence the equation (39) is given. We take no account of the overall phase coming from the determination of the branch. The contour having two arrows, means a double line integral respect to two variable as keeping these ordering.

Next useful contour integral is composed of two kinds of vertex operators V_1 and V_2 , and two kinds of screening charges Q_1 and Q_2 as below.

$$I(z, 0) = [V_1(z)Q_1(z), [Q_2(z), V_2(0)]] \quad (42)$$

The contour lines for Q_1 and Q_2 are deformed to one line having two arrows:

$$\begin{array}{c} 1 \\ \curvearrowright \\ 2 \bullet \\ 0 \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} = (1 - q^{-1}) \begin{array}{c} 1 \\ \curvearrowright \\ 2 \bullet \\ z \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} = (1 - q^{-1})^2 \begin{array}{c} 1 \\ \curvearrowright \\ 2 \bullet \\ z \end{array} \begin{array}{c} 1 \\ \curvearrowright \\ z \end{array} \quad (43)$$

$I(z, 0)$ commutes with both screening charges Q_1 and Q_2 .

$$[Q_i(z_1), I(z, 0)] = 0, \quad (44)$$

This commutativity is concerned by a procedure similar to (41):

$$\text{Diagram (45)} \quad (45)$$

There are another type of integrals, which also commutes with both screening charges $Q_1(z)$ and $Q_2(z)$.

$$[Q_i(z), I_1(z_1, z_2, z_3)] = 0. \quad (46)$$

This commutativity is concerned by a procedure similar to (41) too.

The contour of $I_1(z_1, z_2, z_3)$ can be drawn as below.

$$\text{Diagram (47)} \quad (47)$$

In these diagrams, the contour of Q_2 goes from the position of one screen current s_1 to the position of another screen current s_1 . By mutually replacing the labels 1 and 2 in (47), we obtain $I_2(z_1, z_2, z_3)$.

The identity (35) can be shown by applying Cauchy's integral theorem to the following contours corresponding to (35).

$$\text{Diagram (48)} \quad (48)$$

while we have omitted the labels near the arrows in (48), we assume that the labels 1, 2 can be assigned according to following rules.

$$1 \bullet \xleftarrow{1} \bullet 1, \quad 2 \bullet \xleftarrow{2} \bullet 2, \quad 2 \bullet \xleftarrow{2} \xleftarrow{1} \bullet 1. \quad (49)$$

2.4 exchange relations

In this section, we list the concrete expressions of exchange relations between the six screened vertex operators (22). These calculations are performed by comparing the contours in both side. Then we use the identities introduced in previous subsection.

exchange relations between $\{U^{(1,0)}, U^{(-1,1)}, U^{(0,-1)}\}$

$$\begin{aligned}
U^{(1,0)}(z)U^{(1,0)}(w)|\mu\rangle &= U^{(1,0)}(w)U^{(1,0)}(z)|\mu\rangle, \\
U^{(1,0)}(z)U^{(-1,1)}(w)|\mu\rangle &= U^{(-1,1)}(w)U^{(1,0)}(z)|\mu\rangle + \frac{1}{m+1}\langle\psi|U^{(1,0)}(w)U^{(-1,1)}(z)|\mu\rangle \\
U^{(-1,1)}(z)U^{(1,0)}(w)|\mu\rangle &= \frac{m(m+2)}{(m+1)^2}U^{(1,0)}(w)U^{(-1,1)}(z)|\mu\rangle - \frac{1}{m+1}U^{(-1,1)}(w)U^{(1,0)}(z)|\mu\rangle \\
U^{(-1,1)}(z)U^{(-1,1)}(w)|\mu\rangle &= U^{(-1,1)}(w)U^{(-1,1)}(z)|\mu\rangle \\
U^{(1,0)}(z)U^{(0,-1)}(w)|\mu\rangle &= U^{(0,-1)}(w)U^{(1,0)}(z)|\mu\rangle + \frac{1}{m+n+2}U^{(1,0)}(w)U^{(0,-1)}(z)|\mu\rangle \\
U^{(0,-1)}(z)U^{(1,0)}(w)|\mu\rangle &= \frac{(m+n+1)(m+n+3)}{(m+n+2)^2}U^{(1,0)}(w)U^{(0,-1)}(z)|\mu\rangle \\
&\quad - \frac{1}{m+n+2}U^{(0,-1)}(w)U^{(1,0)}(z)|\mu\rangle \\
U^{(0,-1)}(z)U^{(-1,1)}(w)|\mu\rangle &= U^{(-1,1)}(w)U^{(0,-1)}(z)|\mu\rangle - \frac{1}{n+1}U^{(0,-1)}(w)U^{(-1,1)}(z)|\mu\rangle \\
U^{(-1,1)}(z)U^{(0,-1)}(w)|\mu\rangle &= \frac{n(n+2)}{(n+1)^2}U^{(0,-1)}(w)U^{(-1,1)}(z)|\mu\rangle + \frac{1}{n+1}U^{(-1,1)}(w)U^{(0,-1)}(z)|\mu\rangle \\
U^{(0,-1)}(z)U^{(0,-1)}(w)|\mu\rangle &= U^{(0,-1)}(w)U^{(0,-1)}(z)|\mu\rangle
\end{aligned} \tag{50}$$

exchange relations between $\{U^{(0,1)}, U^{(1,-1)}, U^{(-1,0)}\}$

$$\begin{aligned}
U^{(0,1)}(z)U^{(0,1)}(w)|\mu\rangle &= U^{(0,1)}(w)U^{(0,1)}(z)|\mu\rangle, \\
U^{(0,1)}(z)U^{(1,-1)}(w)|\mu\rangle &= U^{(1,-1)}(w)U^{(0,1)}(z)|\mu\rangle + \frac{1}{n+1}U^{(0,1)}(w)U^{(1,-1)}(z)|\mu\rangle \\
U^{(1,-1)}(z)U^{(0,1)}(w)|\mu\rangle &= \frac{n(n+2)}{(n+1)^2}U^{(0,1)}(w)U^{(1,-1)}(z)|\mu\rangle - \frac{1}{n+1}U^{(1,-1)}(w)U^{(0,1)}(z)|\mu\rangle \\
U^{(1,-1)}(z)U^{(1,-1)}(w)|\mu\rangle &= U^{(1,-1)}(w)U^{(1,-1)}(z)|\mu\rangle \\
U^{(0,1)}(z)U^{(-1,0)}(w)|\mu\rangle &= U^{(-1,0)}(w)U^{(0,1)}(z)|\mu\rangle + \frac{1}{m+n+2}U^{(1,0)}(w)U^{(-1,0)}(z)|\mu\rangle \\
U^{(-1,0)}(z)U^{(0,1)}(w)|\mu\rangle &= \frac{(m+n+1)(m+n+3)}{(m+n+2)^2}U^{(0,1)}(w)U^{(-1,0)}(z)|\mu\rangle \\
&\quad - \frac{1}{m+n+2}U^{(-1,0)}(w)U^{(0,1)}(z)|\mu\rangle \\
U^{(-1,0)}(z)U^{(1,-1)}(w)|\mu\rangle &= U^{(1,-1)}(w)U^{(-1,0)}(z)|\mu\rangle - \frac{1}{m+1}U^{(-1,0)}(w)U^{(1,-1)}(z)|\mu\rangle \\
U^{(1,-1)}(z)U^{(-1,0)}(w)|\mu\rangle &= \frac{m(m+2)}{(m+1)^2}U^{(-1,0)}(w)U^{(1,-1)}(z)|\mu\rangle + \frac{1}{m+1}U^{(1,-1)}(w)U^{(-1,0)}(z)|\mu\rangle \\
U^{(-1,0)}(z)U^{(-1,0)}(w)|\mu\rangle &= U^{(-1,0)}(w)U^{(-1,0)}(z)|\mu\rangle
\end{aligned} \tag{51}$$

Up to now we have given the exchange relations between $\{U^{(1,0)}, U^{(-1,1)}, U^{(0,-1)}\}$ and ones between $\{U^{(0,1)}, U^{(1,-1)}, U^{(-1,0)}\}$ with closed form, respectively.

exchange relations between $\{U^{(1,0)}, U^{(-1,1)}, U^{(0,-1)}\}$ and $\{U^{(0,1)}, U^{(1,-1)}, U^{(-1,0)}\}$

$$\begin{aligned}
U^{(0,1)}(z)U^{(1,0)}(w)|\mu\rangle &= U^{(1,0)}(w)U^{(0,1)}(z)|\mu\rangle, \\
U^{(1,-1)}(z)U^{(1,0)}(w)|\mu\rangle &= U^{(1,0)}(w)U^{(1,-1)}(z)|\mu\rangle, \\
U^{(0,1)}(z)U^{(-1,1)}(w)|\mu\rangle &= U^{(-1,1)}(w)U^{(0,1)}(z)|\mu\rangle, \\
U^{(-1,0)}(z)U^{(1,0)}(w)|\mu\rangle &= \frac{m(m+2)(m+n+1)(m+n+3)}{(m+1)^2(m+n+2)^2}U^{(1,0)}(w)U^{(-1,0)}(z)|\mu\rangle \\
&\quad + \frac{m+2}{(m+1)(m+n+2)}U^{(0,-1)}(w)U^{(0,1)}(z)|\mu\rangle, \\
&\quad - \frac{n(m+n+3)}{(m+1)(n+1)(m+n+2)}U^{(-1,1)}(w)U^{(1,-1)}(z)|\mu\rangle \\
U^{(1,-1)}(z)U^{(-1,1)}(w)|\mu\rangle &= \frac{n}{n+1}U^{(-1,1)}(w)U^{(1,-1)}(z)|\mu\rangle - \frac{1}{n+1}U^{(0,-1)}(w)U^{(0,1)}(z)|\mu\rangle \\
&\quad + \frac{(n+2)(m+n+1)}{(m+1)(n+1)(m+n+2)}U^{(1,0)}(w)U^{(-1,0)}(z)|\mu\rangle \\
U^{(0,1)}(z)U^{(0,-1)}(w)|\mu\rangle &= U^{(0,-1)}(w)U^{(0,1)}(z)|\mu\rangle + \frac{1}{n+1}U^{(-1,1)}(w)U^{(1,-1)}(z)|\mu\rangle \\
&\quad - \frac{m}{(m+1)(m+n+2)}U^{(1,0)}(w)U^{(-1,0)}(z)|\mu\rangle \\
U^{(1,-1)}(z)U^{(0,-1)}(w)|\mu\rangle &= U^{(0,-1)}(w)U^{(1,-1)}(z)|\mu\rangle \\
U^{(0,-1)}(z)U^{(1,-1)}(w)|\mu\rangle &= U^{(1,-1)}(w)U^{(0,-1)}(z)|\mu\rangle \\
U^{(0,-1)}(z)U^{(-1,0)}(w)|\mu\rangle &= U^{(-1,0)}(w)U^{(0,-1)}(z)|\mu\rangle.
\end{aligned} \tag{52}$$

So we have demonstrated that the exchange relations between the six bases(22), can be expressible in closed form

3 spider diagrams

In this section, we give a brief review of the spider diagrams considered in [1, 2]. We take a limit $q \rightarrow 1$ to obtain semiclassical limit. In this case, the fundamental relations for the spider diagrams are defined as following.

$$\begin{aligned}
\text{Clockwise circle} &= \text{Counter-clockwise circle} = 3, \quad \text{Circle with two arrows pointing out} = -2 \text{ two parallel arrows}, \\
\text{Spider diagram with four legs} &= \text{Two crossing lines} + \text{Two crossing lines with different orientation}
\end{aligned} \tag{53}$$

It is defined that the crossing lines can be decomposed into two diagrams.

$$\text{Crossing lines} = \text{Two parallel lines with arrows} + \text{Spider diagram with four legs} \tag{54}$$

The lines crossing twice come untied.

$$\begin{aligned}
\text{Diagram 1} &= \text{Diagram 2} + 2 \text{Diagram 3} + \text{Diagram 4} = \text{Diagram 5} \\
\text{Diagram 6} &= \text{Diagram 7} + \text{Diagram 8} + \text{Diagram 9} + \text{Diagram 10} = \text{Diagram 11}
\end{aligned} \tag{55}$$

There are diagrams, called *clasp*, which possess property of the projection operator as following.

$$\begin{aligned}
\text{Diagram 12} &\equiv \frac{1}{2} \left[\text{Diagram 13} + \text{Diagram 14} \right] = \text{Diagram 15} + \frac{1}{2} \text{Diagram 16} , \\
\text{Diagram 17} &\equiv \text{Diagram 18} - \frac{1}{3} \text{Diagram 19} .
\end{aligned} \tag{56}$$

$$\begin{aligned}
\text{Diagram 20} &= \text{Diagram 21} , \\
\text{Diagram 22} &= \text{Diagram 23} .
\end{aligned} \tag{57}$$

In general, one can introduce the clasps having much more external lines. For simplification, we represent the internal clasps as below.

$$\frac{\uparrow}{n \uparrow} := \underbrace{\frac{\uparrow \uparrow \uparrow \uparrow \dots \uparrow}{\uparrow \uparrow \uparrow \uparrow \dots \uparrow}}_n , \quad \frac{\uparrow}{\mu \uparrow} := \frac{\uparrow}{m \uparrow} \frac{\downarrow}{n \downarrow} . \tag{58}$$

where $\mu = (m, n)$. In this paper, the internal clasp satisfies the equations.

$$\text{Diagram 24} = \text{Diagram 25} = 0 . \tag{59}$$

The clasps are obtained by the mathematical inductions.

$$\begin{aligned}
\frac{\uparrow}{n \uparrow} &= \frac{\uparrow}{n-1 \uparrow} \uparrow + \frac{n-1}{n} \text{Diagram 26} \\
\frac{\uparrow}{\mu \uparrow} &= \sum_{k=0}^{\min(m,n)} (-1)^k \frac{m!n!(m+n-k+1)}{(m-k)!(n-k)!(m+n+1)!k!} \text{Diagram 27}
\end{aligned} \tag{60}$$

In this paper, we use only first two terms of the expansion (60), because of that we consider only simple spider diagrams like as (62).

$$\frac{\uparrow}{\mu \uparrow} = \frac{\uparrow}{m \uparrow} \frac{\downarrow}{n \downarrow} - \frac{mn}{m+n+1} \text{Diagram 28} + \dots \tag{61}$$

$(1, 0)$, $(-1, 1)$ and $(0, -1)$.

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 1} \end{array} \\ \begin{array}{c} m+2 \\ n \end{array} \end{array} = \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 2} \end{array} \\ \begin{array}{c} m+2 \\ n \end{array} \end{array} \quad (64)$$

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 3} \end{array} \\ \begin{array}{c} m \\ n+1 \end{array} \end{array} = \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 4} \end{array} \\ \begin{array}{c} m \\ n+1 \end{array} \end{array} + \frac{1}{m+1} \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 5} \end{array} \\ \begin{array}{c} m \\ n+1 \end{array} \end{array} \quad (65)$$

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 6} \end{array} \\ \begin{array}{c} m \\ n+1 \end{array} \end{array} = \frac{m(m+2)}{(m+1)^2} \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 7} \end{array} \\ \begin{array}{c} m \\ n+1 \end{array} \end{array} - \frac{1}{m+1} \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 8} \end{array} \\ \begin{array}{c} m \\ n+1 \end{array} \end{array} \quad (66)$$

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 9} \end{array} \\ \begin{array}{c} m-2 \\ n+2 \end{array} \end{array} = \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 10} \end{array} \\ \begin{array}{c} m-2 \\ n+2 \end{array} \end{array} \quad (67)$$

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 11} \end{array} \\ \begin{array}{c} m+1 \\ n-1 \end{array} \end{array} = \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 12} \end{array} \\ \begin{array}{c} m+1 \\ n-1 \end{array} \end{array} - \frac{1}{m+n+2} \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 13} \end{array} \\ \begin{array}{c} m+1 \\ n-1 \end{array} \end{array} \quad (68)$$

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 14} \end{array} \\ \begin{array}{c} m+1 \\ n-1 \end{array} \end{array} = \frac{(m+n+1)(m+n+3)}{(m+n+2)^2} \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 15} \end{array} \\ \begin{array}{c} m+1 \\ n-1 \end{array} \end{array} - \frac{1}{m+n+2} \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 16} \end{array} \\ \begin{array}{c} m+1 \\ n-1 \end{array} \end{array} \quad (69)$$

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 17} \end{array} \\ \begin{array}{c} m-1 \\ n \end{array} \end{array} = \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 18} \end{array} \\ \begin{array}{c} m-1 \\ n \end{array} \end{array} + \frac{1}{n+1} \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 19} \end{array} \\ \begin{array}{c} m-1 \\ n \end{array} \end{array} \quad (70)$$

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 20} \end{array} \\ \begin{array}{c} m-1 \\ n \end{array} \end{array} = \frac{n(n+2)}{(n+1)^2} \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 21} \end{array} \\ \begin{array}{c} m-1 \\ n \end{array} \end{array} - \frac{1}{n+1} \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 22} \end{array} \\ \begin{array}{c} m-1 \\ n \end{array} \end{array} \quad (71)$$

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 23} \end{array} \\ \begin{array}{c} m \\ n-2 \end{array} \end{array} = \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 24} \end{array} \\ \begin{array}{c} m \\ n-2 \end{array} \end{array} \quad (72)$$

The above coefficients correspond to the ones in (50).

The following equations are the exchange relations between spider diagrams(62) having the weights

$(0, 1)$, $(1, -1)$ and $(-1, 0)$.

[illegible]

$$\begin{array}{c} \text{Diagram 1} \end{array} = \begin{array}{c} \text{Diagram 2} \end{array} + \frac{1}{n+1} \begin{array}{c} \text{Diagram 3} \end{array} \quad (74)$$

$$\begin{aligned}
& \text{Diagram 1} = \frac{n(n+2)}{(n+1)^2} \text{Diagram 2} - \frac{1}{n+1} \text{Diagram 3} \quad (75)
\end{aligned}$$

[illegible]

[illegible]

$$\begin{aligned}
& \text{Diagram 1} = \frac{(m+n+1)(m+n+3)}{(m+n+2)^2} \text{Diagram 2} - \frac{1}{m+n+2} \text{Diagram 3} \quad (78)
\end{aligned}$$

$$\begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \begin{array}{c} \text{Diagram 1: A node with two incoming arrows from above, labeled 1 and 1. Below it is a horizontal line with nodes labeled m, n-1, n, m. Arrows point from m to n-1 and from n to m. A curved arrow points from the node below 1 to the node below n.} \end{array} \\ \end{array} = \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \begin{array}{c} \text{Diagram 2: Similar to Diagram 1, but the curved arrow points from the node below 1 to the node below m.} \end{array} + \frac{1}{m+1} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \begin{array}{c} \text{Diagram 3: Similar to Diagram 1, but the curved arrow points from the node below 1 to the node below n.} \end{array} \end{array} \quad (79)$$

$$\begin{array}{c} \text{Diagram 1} \end{array} = \frac{m(m+2)}{(m+1)^2} \begin{array}{c} \text{Diagram 2} \end{array} - \frac{1}{m+1} \begin{array}{c} \text{Diagram 3} \end{array} \quad (80)$$

[illegible]

The above coefficients correspond to the ones in (51).

The following equations are the exchange relations between spider diagrams(62) having the weights

$\{(1, 0), (-1, 1), (0, -1)\}$ and $\{(0, 1), (1, -1), (-1, 0)\}$.

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \begin{array}{c} \text{Diagram 1} \end{array} \end{array} = \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 2} \end{array} \end{array} \quad (82)$$

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 3} \end{array} = \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 4} \end{array} \end{array} \quad (83)$$

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 5} \end{array} = \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 6} \end{array} \end{array} \quad (84)$$

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 7} \end{array} = \frac{m(m+2)(m+n+1)(m+n+3)}{(m+1)^2(m+n+2)^2} \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 8} \end{array} \end{array} + \frac{m+2}{(m+1)(m+n+2)} \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 9} \end{array} \end{array} \\ - \frac{n(m+n+3)}{(m+1)(n+1)(m+n+2)} \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 10} \end{array} \end{array} \quad (85)$$

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 11} \end{array} = \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 12} \end{array} \end{array} + \frac{1}{n+1} \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 13} \end{array} \end{array} - \frac{m}{(m+1)(m+n+2)} \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 14} \end{array} \end{array} \quad (86)$$

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 15} \end{array} = \frac{n}{n+1} \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 16} \end{array} \end{array} - \frac{1}{n+1} \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 17} \end{array} \end{array} + \frac{(n+2)(m+n+1)}{(m+1)(n+1)(m+n+2)} \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 18} \end{array} \end{array} \quad (87)$$

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 19} \end{array} = \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 20} \end{array} \end{array} \quad (88)$$

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 21} \end{array} = \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 22} \end{array} \end{array} \quad (89)$$

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 23} \end{array} = \begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \downarrow \quad \downarrow \\ \text{Diagram 24} \end{array} \end{array} \quad (90)$$

$$(91)$$

The above coefficients correspond to the ones in (52).

4 correspondence

In the previous two sections, we have calculated the exchange relations in both the sl_3 WZNW model and the spider diagram sides in semiclassical limit. As a result, there is the correspondence between the two coefficients as following form.

$$[c_{\text{WZNW}}(\mu)]_{\lambda_3 \lambda_4}^{\lambda_1 \lambda_2} = [c_{\text{spider}}(\mu)]_{\lambda_3 \lambda_4}^{\lambda_1 \lambda_2}. \quad (92)$$

These coefficients are defined in (27) and (63). In appendix A, we give the intermediate expressions of spider diagrams.

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A calculations of spider diagrams

In this appendix, we simplify both twisted spider diagrams and untwisted ones to compare each other. The simplification is performed by expanding the central clasps according to (61) and by using the fundamental equations (53).

All the twisted spider diagrams can be expressible in terms of untwisted ones finally. These results are listed in Sec.3.1.

weight $(m, n) \rightarrow (m + 2, n)$

The twisted diagram is

$$\begin{array}{c} \begin{array}{c} \downarrow 1 \quad \downarrow 1 \\ \text{twisted spider diagram} \end{array} \\ \begin{array}{c} m+2 \quad n \end{array} \end{array} = \begin{array}{c} \begin{array}{c} \downarrow 1 \quad \downarrow 1 \\ \text{untwisted spider diagram} \end{array} \\ \begin{array}{c} m+2 \quad n \end{array} \end{array} \quad (93)$$

The untwisted diagram is

$$\begin{array}{c} \begin{array}{c} \downarrow 1 \quad \downarrow 1 \\ \text{untwisted spider diagram} \end{array} \\ \begin{array}{c} m+2 \quad n \end{array} \end{array} = \begin{array}{c} \begin{array}{c} \downarrow 1 \quad \downarrow 1 \\ \text{twisted spider diagram} \end{array} \\ \begin{array}{c} m+2 \quad n \end{array} \end{array} \quad (94)$$

weight $(m, n) \rightarrow (m, n + 1)$

The twisted diagrams are

$$\begin{array}{c} \begin{array}{c} \downarrow 1 \quad \downarrow 1 \\ \text{twisted spider diagram} \end{array} \\ \begin{array}{c} m \quad n+1 \end{array} \end{array} = \begin{array}{c} \begin{array}{c} \downarrow 1 \quad \downarrow 1 \\ \text{untwisted spider diagram} \end{array} \\ \begin{array}{c} m \quad n+1 \end{array} \end{array} \\ \\ = \begin{array}{c} \begin{array}{c} \downarrow 1 \quad \downarrow 1 \\ \text{untwisted spider diagram} \end{array} \\ \begin{array}{c} m \quad n+1 \end{array} \end{array} + \begin{array}{c} \begin{array}{c} \downarrow 1 \quad \downarrow 1 \\ \text{untwisted spider diagram} \end{array} \\ \begin{array}{c} m \quad n+1 \end{array} \end{array} \quad (95)$$

$$\begin{array}{c} \begin{array}{c} \downarrow 1 \quad \downarrow 1 \\ \text{twisted spider diagram} \end{array} \\ \begin{array}{c} m \quad n+1 \end{array} \end{array} = \frac{m}{m+1} \begin{array}{c} \begin{array}{c} \downarrow 1 \quad \downarrow 1 \\ \text{untwisted spider diagram} \end{array} \\ \begin{array}{c} m \quad n+1 \end{array} \end{array} - \frac{1}{m+1} \begin{array}{c} \begin{array}{c} \downarrow 1 \quad \downarrow 1 \\ \text{untwisted spider diagram} \end{array} \\ \begin{array}{c} m+1 \quad n \end{array} \end{array} \quad (96)$$

The untwisted diagrams are

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \quad (97)$$

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \frac{1}{m+1} \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} + \frac{m}{m+1} \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \quad (98)$$

weight $(m, n) \rightarrow (m+1, n-1)$

The twisted diagrams are

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} - \frac{1}{m+n+2} \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \quad (99)$$

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \quad (100)$$

The untwisted diagrams are

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} - \frac{1}{m+n+2} \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \quad (101)$$

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} \quad (102)$$

weight $(m, n) \rightarrow (m-1, n)$

The twisted diagrams are

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \\ \text{Diagram 5} \end{array} \quad (103)$$

$$\begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = \frac{n}{n+1} \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} - \frac{1}{n+1} \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} \quad (104)$$

The untwisted diagrams are

$$\begin{aligned}
\text{Diagram 1} &= \frac{n}{n+1} \text{Diagram 2} + \frac{1}{n+1} \text{Diagram 3} \\
&= \text{Diagram 4} + \frac{n}{n+1} \text{Diagram 5}
\end{aligned} \tag{105}$$

$$\begin{array}{c} \text{\tiny 1} \\ \downarrow \\ \text{\tiny 1} \\ \downarrow \\ m-1 \quad n \end{array} = \begin{array}{c} \text{\tiny 1} \\ \downarrow \\ \text{\tiny 1} \\ \downarrow \\ m-1 \quad n \end{array} \quad (106)$$

weight $(m, n) \rightarrow (m, n - 2)$

The twisted diagram is

[illegible]

The untwisted diagram is

[illegible]

weight $(m, n) \rightarrow (m - 2, n + 2)$

The twisted diagram is

[illegible]

The untwisted diagram is

$$\begin{array}{c}
\begin{array}{ccccc}
1 & & 1 & & \\
\downarrow & & \downarrow & & \\
\begin{array}{ccccc}
\leftarrow & \leftarrow & \leftarrow & \leftarrow & \leftarrow \\
\hline
m-2 & & m-2 & & m \\
\hline
n+2 & & n+2 & & n
\end{array}
\end{array}
=
\begin{array}{c}
\begin{array}{ccccc}
1 & & 1 & & \\
\downarrow & & \downarrow & & \\
\begin{array}{ccccc}
\leftarrow & \leftarrow & \leftarrow & \leftarrow & \leftarrow \\
\hline
m-2 & & m-2 & & m \\
\hline
n+2 & & n+2 & & n
\end{array}
\end{array}
\end{array} \quad (110)$$

weight $(m, n) \rightarrow (m, n + 2)$

The twisted diagram is

$$\begin{array}{c}
\begin{array}{c}
1 \quad 1 \\
\downarrow \quad \downarrow \\
\begin{array}{c}
\text{Diagram 1: A box with } m \text{ inputs on the left and } m \text{ outputs on the right. Two inputs are labeled } n+2 \text{ and } n. \end{array}
\end{array}
=
\begin{array}{c}
\begin{array}{c}
1 \quad 1 \\
\downarrow \quad \downarrow \\
\begin{array}{c}
\text{Diagram 2: A box with } m \text{ inputs on the left and } m \text{ outputs on the right. Two inputs are labeled } n+2 \text{ and } n. \end{array}
\end{array}
\end{array}
\quad (111)$$

The untwisted diagram is

$$\begin{array}{c} \text{\tiny 1} \\ \vdots \\ \text{\tiny 1} \end{array} \quad \begin{array}{c} \text{\tiny 1} \\ \vdots \\ \text{\tiny 1} \end{array} \quad \begin{array}{c} m \\ \leftarrow \\ n+2 \end{array} \quad \begin{array}{c} \leftarrow \\ \vdots \\ \leftarrow \end{array} \quad \begin{array}{c} m \\ \rightarrow \\ n \end{array} = \begin{array}{c} \text{\tiny 1} \\ \vdots \\ \text{\tiny 1} \end{array} \quad \begin{array}{c} m \\ \leftarrow \\ n+2 \end{array} \quad \begin{array}{c} \leftarrow \\ \vdots \\ \leftarrow \end{array} \quad \begin{array}{c} m \\ \rightarrow \\ n \end{array} \quad (112)$$

weight $(m, n) \rightarrow (m + 1, n)$

The twisted diagrams are

$$\begin{aligned}
 \text{Diagram 1} &= \text{Diagram 2} \\
 &= \text{Diagram 3} + \text{Diagram 4}
 \end{aligned} \tag{113}$$

$$\text{Diagram 5} = \frac{n}{n+1} \text{Diagram 6} - \frac{1}{n+1} \text{Diagram 7} \tag{114}$$

The untwisted diagrams are

$$\text{Diagram 8} = \text{Diagram 9} \tag{115}$$

$$\begin{aligned}
 \text{Diagram 10} &= \frac{1}{n+1} \text{Diagram 11} + \frac{n}{n+1} \text{Diagram 12} \\
 &= \text{Diagram 13} + \frac{n}{n+1} \text{Diagram 14}
 \end{aligned} \tag{116}$$

weight $(m, n) \rightarrow (m + 2, n - 2)$

The twisted diagram is

$$\text{Diagram 15} = \text{Diagram 16} \tag{117}$$

The untwisted diagram is

$$\text{Diagram 17} = \text{Diagram 18} \tag{118}$$

weight $(m, n) \rightarrow (m - 1, n + 1)$

The twisted diagrams are

$$\text{Diagram 19} = \text{Diagram 20} - \frac{1}{m+n+2} \text{Diagram 21} \tag{119}$$

$$\text{Diagram 22} = \text{Diagram 23} \tag{120}$$

The untwisted diagrams are

[illegible]

$$\begin{array}{c} \begin{array}{c} 1 \quad 1 \\ \uparrow \quad \uparrow \\ m-1 \quad m \\ \leftarrow \quad \leftarrow \\ n+1 \quad n \end{array} \\ = \quad \begin{array}{c} 1 \quad 1 \\ \uparrow \quad \uparrow \\ m-1 \quad m \\ \leftarrow \quad \leftarrow \\ n+1 \quad n \end{array} \end{array} \quad (122)$$

weight $(m, n) \rightarrow (m, n - 1)$

The twisted diagrams are

$$\begin{array}{c}
\begin{array}{c} \text{Diagram 1: } \text{A node with two incoming arrows from above labeled 1, and two outgoing arrows to the left and right labeled } m \text{ and } n \text{ respectively.} \\ \text{Diagram 2: } \text{A node with two incoming arrows from above labeled 1, and two outgoing arrows to the left and right labeled } m \text{ and } n \text{ respectively.} \end{array} \\
= \\
\begin{array}{c} \text{Diagram 3: } \text{A node with two incoming arrows from above labeled 1, and two outgoing arrows to the left and right labeled } m \text{ and } n \text{ respectively.} \\ \text{Diagram 4: } \text{A node with two incoming arrows from above labeled 1, and two outgoing arrows to the left and right labeled } m \text{ and } n \text{ respectively.} \end{array} \\
= \\
\begin{array}{c} \text{Diagram 5: } \text{A node with two incoming arrows from above labeled 1, and two outgoing arrows to the left and right labeled } m \text{ and } n \text{ respectively.} \\ \text{Diagram 6: } \text{A node with two incoming arrows from above labeled 1, and two outgoing arrows to the left and right labeled } m \text{ and } n \text{ respectively.} \end{array}
\end{array} \quad (123)$$

$$\begin{array}{c} \text{\tiny 1} & \text{\tiny 1} \\ \nearrow & \nwarrow \\ \text{\tiny } n-1 & \text{\tiny } m \\ \leftarrow & \rightarrow \\ \text{\tiny } n-1 & \text{\tiny } n \end{array} = \frac{m}{m+1} \begin{array}{c} \text{\tiny 1} & \text{\tiny 1} \\ \nearrow & \nwarrow \\ \text{\tiny } n-1 & \text{\tiny } m \\ \leftarrow & \rightarrow \\ \text{\tiny } n-1 & \text{\tiny } n \end{array} - \frac{1}{m+1} \begin{array}{c} \text{\tiny 1} & \text{\tiny 1} \\ \curvearrowright & \nwarrow \\ \text{\tiny } n-1 & \text{\tiny } m \\ \leftarrow & \rightarrow \\ \text{\tiny } n-1 & \text{\tiny } n \end{array} \quad (124)$$

The untwisted diagrams are

$$\begin{aligned}
& \text{Diagram 1} = \frac{m}{m+1} \text{Diagram 2} + \frac{1}{m+1} \text{Diagram 3} \\
& \text{Diagram 1} = \text{Diagram 4} + \frac{m}{m+1} \text{Diagram 5}
\end{aligned} \tag{125}$$

$$\begin{array}{c} \text{\tiny 1} \\ \uparrow \\ \text{\tiny 1} \\ \uparrow \\ \text{\tiny m} \quad \text{\tiny n-1} \end{array} = \begin{array}{c} \text{\tiny 1} \\ \uparrow \\ \text{\tiny 1} \\ \uparrow \\ \text{\tiny m} \quad \text{\tiny n-1} \end{array} \quad (126)$$

weight $(m, n) \rightarrow (m - 2, n)$

The twisted diagram is

[illegible]

The untwisted diagram is

[illegible]

The untwisted diagram is

Diagrammatic equation (142) shows two equivalent configurations of strands. On the left, two strands labeled 1 enter from the top, connect to a horizontal strand labeled $m-2$ on the left and m on the right, which then connects to a strand labeled $n+1$ on the left and n on the right. On the right, the strands are rearranged to show a different topological configuration. The equation is labeled (142).

weight $(m, n) \rightarrow (m + 1, n - 2)$

The twisted diagram is

Diagrammatic equation (143) shows two equivalent configurations of strands. On the left, two strands labeled 1 enter from the top, connect to a horizontal strand labeled $m+1$ on the left and n on the right, which then connects to a strand labeled $n-2$ on the left and n on the right. On the right, the strands are rearranged to show a different topological configuration. The equation is labeled (143).

The untwisted diagram is

Diagrammatic equation (144) shows two equivalent configurations of strands. On the left, two strands labeled 1 enter from the top, connect to a horizontal strand labeled $m+1$ on the left and n on the right, which then connects to a strand labeled $n-2$ on the left and n on the right. On the right, the strands are rearranged to show a different topological configuration. The equation is labeled (144).

weight $(m, n) \rightarrow (m - 1, n - 1)$

The twisted diagram is

Diagrammatic equation (145) shows two equivalent configurations of strands. On the left, two strands labeled 1 enter from the top, connect to a horizontal strand labeled $m-1$ on the left and n on the right, which then connects to a strand labeled $n-1$ on the left and n on the right. On the right, the strands are rearranged to show a different topological configuration. The equation is labeled (145).

The untwisted diagram is

Diagrammatic equation (146) shows two equivalent configurations of strands. On the left, two strands labeled 1 enter from the top, connect to a horizontal strand labeled $m-1$ on the left and n on the right, which then connects to a strand labeled $n-1$ on the left and n on the right. On the right, the strands are rearranged to show a different topological configuration. The equation is labeled (146).

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